The failure of Mirels' theory to give better results is ascribed to restrictions inherent in his local similarity approach. Only perturbations in u_2 are taken into account, whereas variations in U_s, p_2, and ρ_2 are neglected.

Remaining discrepancies may be explained by other parameters not included in either of the two theories, for example, the diaphragm opening mechanism.

References

[1] Roshko, A. and Smith, J. A., "Measurements of test time in the GALCIT 17-inch shock tube," AIAA J. **2**, 186–187 (1964).

[2] Mirels, H., "Shock tube test time limitation due to turbulent-wall boundary layer," AIAA J. **2**, 84–93 (1964).

[3] Brocher, E. F., "Hot flow length and testing time in real shock tube flow," Phys. Fluids **7**, 347 (1964).

Inadequacy of Nodal Connections in a Stiffness Solution for Plate Bending

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In choosing element displacement functions for a stiffness method of analysis the following criteria must be met:

1) It must be possible to represent the rigid body motions of an element. Otherwise the equilibrium conditions of the element as a whole are falsified.[1]

2) It must be possible to represent states of constant stress. Otherwise, as the mesh of elements is finely subdivided, there is no guarantee that the stresses will converge toward continuous functions; in general, they will not converge at all.[2]

3) Where neighboring elements abut between nodes, there must be no discontinuity of slopes and deflections between the two elements.[3] Otherwise the idealization includes hinges or sawcuts between elements as well as the constraints imposed by the displacement functions. Therefore the bound theorems no longer hold[1, 2, 4] and the solution cannot be described as "pure stiffness."

This note shows that 2 and 3 are incompatible for plate elements in bending, which implies that elements should not be tied at nodes in plate and shell problems but should be matched along boundaries.[5]

Triangle ABC in Fig. 1 is given unit rate of twist about AB, so that at every point $w = xy$. Thus the quantity known as "Torsion" $= \partial^2 w/\partial x \partial y = w_{xy}$, is unity at every point in ABC. It is an easy matter to calculate the nodal rotations θ_1, θ_2, and θ_3 in the directions of the sides, and the nodal deflection δ. At least one of these, applied alone, must give nonzero w_{xy} at A. Say it is δ, θ_2, or θ_3 (or a combination of them) which gives $w_{xy} = K \neq 0$. Consider any undistorted triangle ABC', which is tied to ABC distorted at C. Near A, in ABC

$$\frac{\partial w}{\partial y} = \frac{\partial^2 w}{\partial x \partial y} \times dx = K\,(dx) \qquad (1)$$

whereas in ABC' it is zero. The nonconformity of slope, required to be zero along AB, apparently has a nonzero derivative along AB; this is a contradiction.

Suppose, however, it is θ_1 that gives $w_{xy} = K \neq 0$ at A. Let $AC'B$ be a reflection of ACB about AB, and let both be distorted by θ_1 only. By symmetry, if the deflections on AB are to conform they must be zero, i.e., AB cannot move. Also AC cannot move, otherwise an undistorted triangle ADC could not conform. It follows that, near A, $w = K_1 \times$ (distance from AB) $\times$ (distance from AC) where K_1 is chosen to give the xy term a coefficient K. It follows as before that slope conformity with triangle ADC is impossible.

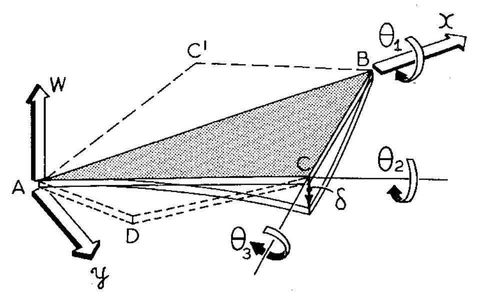

Fig. 1. Illustration of torsional deformation of element ABC with attached triangle.

These arguments extend without difficulty to the polygonal element. If the displacement functions have singularities at the nodes, such that w_{xy} has no unique limiting value, this proof fails.

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References

[1] Melosh, R. J., "Basis for derivation of matrices for the direct stiffness method," AIAA J. **1**, 1631–1637 (1963).

[2] Irons, B. and Barlow, J., "Comment on 'Matrices for the direct stiffness method,'" AIAA J. **2**, 403 (1964).

[3] Pian, T. H. H., "Derivation of element stiffness matrices by assumed stress distributions," AIAA J. **2**, 1333–1336 (1964).

[4] Genot, A., "Bornes aux coefficients d'influence," Rev. Universelle Mines **17**, 315–326 (May 1961).

[5] Jones, R. E., "A generalization of the direct stiffness method of structural analysis," AIAA J. **2**, 821–826 (1964).

Structural Eigenvalue Problems: Elimination of Unwanted Variables

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The tendency in structural analysis is to use hundreds or thousands of variables, whereas the processes involved in finding eigenvalues of full matrices favor tens of variables, or at most slightly over 100. In frequency calculations the classic (but inefficient) technique is to use discrete masses associated with certain selected deflections. A better technique in simple cases is to use fewer elements and to write the kinetic energy and the strain energy in terms of the same assumed deflected shape, with distributed mass. However, engineers do not usually divide a structure into many elements if it can be avoided.

The proposed method uses distributed mass in the KE but retains only a small proportion of the nodal deflections, hereafter termed "masters." The remaining "slave" deflections take values giving least strain energy, regardless of what this does to the KE. Thus a slave node is assumed free from inertial forces. The argument is most clearly visualized in the case of a cantilever. If one takes a result from a discrete mass calculation, draws a smooth curve through the deflected points, and recalculates the KE from the curve, the natural frequency can be corrected to give tolerably good answers. The practical engineer may use interpolation formulas or french curves, or he may prefer to use a flexible beam to draw his smooth curve. But the best flexible beam to use would be the cantilever itself, especially if it had dis-

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continuities of section. Reference 1 shows what excellent results this sort of technique can give. The same argument applies to a stability calculation, where the second-order strain energy replaces the KE.

Of nodal deflections that tend to be "slaves," the following are worth mentioning: 1) in a cantilever-like structure, the lengthwise movements, if these are associated with high frequencies; 2) again, movements near the encastered root, because these also contribute little KE in the lower modes; and 3) the slopes: although these are vital variables, they are unlikely to be used in plotting the modal shapes from the eigenvectors.

To trim the eigenvalue calculation to an acceptable size, it may be necessary to retain as masters only each third or fifth of the remaining deflections. Rayleigh's principle (that a first-order error in modal shape causes only a second-order error in frequency) is borne in mind throughout, and one must also remember that unskilled use of the device can suppress an important frequency.

To clarify the numerical procedure, consider the strain and kinetic energies of a system characterized by only three deflections, x_1, x_2, and x_3, of which x_2 becomes a slave:

$$\text{SE} = \tfrac{1}{2}K_{11}x_1^2 + \tfrac{1}{2}K_{22}x_2^2 + \tfrac{1}{2}K_{33}x_3^2 + K_{12}x_1x_2 + K_{23}x_2x_3 + K_{31}x_3x_1$$

$$\text{KE} = \omega^2/g\{\tfrac{1}{2}M_{11}x_1^2 + \tfrac{1}{2}M_{22}x_2^2 + \tfrac{1}{2}M_{33}x_3^2 + M_{12}x_1x_2 + M_{23}x_2x_3 + M_{31}x_3x_1\}$$

Therefore, for minimum SE,

$$K_{21}x_1 + K_{22}x_2 + K_{23}x_3 = 0 \qquad (1)$$

Thus

$$\text{KE} = \frac{\omega^2}{g}\left\{\frac{1}{2}\left[M_{11} - 2M_{12}\left(\frac{K_{21}}{K_{22}}\right) + M_{22}\left(\frac{K_{21}}{K_{22}}\right)^2\right]x_1^2 + \left[M_{13} - M_{23}\left(\frac{K_{21}}{K_{22}}\right) - M_{21}\left(\frac{K_{23}}{K_{22}}\right) + M_{22}\left(\frac{K_{21}}{K_{22}}\right)\times\left(\frac{K_{23}}{K_{22}}\right)\right]x_1x_3 + \frac{1}{2}\left[M_{33} - 2M_{23}\left(\frac{K_{23}}{K_{22}}\right) + M_{22}\left(\frac{K_{23}}{K_{22}}\right)^2\right]x_3^2\right\}$$

In a problem with many variables, the SE and KE are written as follows:

$$\frac{1}{2}[x_1 \ldots x_n][K]\begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \qquad \frac{\omega^2}{2g}[x_1 \ldots x_n][M]\begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$$

When the variable x_s becomes a slave, row and column s are deleted from $[K]$ and $[M]$, and their i, j terms become

$$M_{ij} - M_{is}\left(\frac{K_{sj}}{K_{ss}}\right) - M_{sj}\left(\frac{K_{is}}{K_{ss}}\right) + M_{ss}\left(\frac{K_{sj}}{K_{ss}}\right)\left(\frac{K_{is}}{K_{ss}}\right) \qquad (2)$$

$$K_{ij} - K_{is}(K_{sj}/K_{ss}) \qquad (3)$$

Expression (2) reduces to (3) when K replaces M. Although (2) looks complicated, it is calculated in only two steps, each resembling (3):

$$M_{ij}^* = M_{ij} - M_{is}(K_{sj}/K_{ss})$$

$$M_{ij}^{**} = M_{ij}^* - M_{sj}^*(K_{is}/K_{ss})$$

A more complicated routine, now described, puts an estimated load at a slave point. In the three-point case, for example, if F_i represents the loads and M_i the point masses (as in a discrete mass calculation), and if S represents distance along a beam,

$$(F_2/M_2)(S_3 - S_1) \doteq (F_1/M_1)(S_3 - S_2) + (F_3/M_3)(S_2 - S_1) \qquad (4)$$

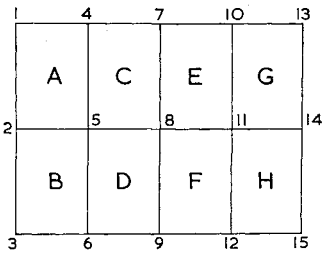

Fig. 1. Diagram of two-abreast mesh of elements.

Writing (4) in terms of K_{ij} gives an expression for x_2. Substituting this into the KE and SE expressions means that the K in (1), and hence the bracketed K ratios in (2), are modified, and (3) must have the same form as (2). When x_2 has become a slave, the mass M_2 must be reallocated for subsequent eliminations, e.g., M_1 becomes

$$M_1 + M_2(S_3 - S_2)/(S_3 - S_1) \qquad (5)$$

If this process is numerically well conditioned, the experience of Ref. 1 suggests that the frequencies should be very good.

The most attractive feature of (2) and (3) is not obvious. Assume two areas $[K]$ and $[M]$ are reserved in core, and the contributions of each element in turn are added in. Under certain conditions, most of the variables can be eliminated after each adding-in operation to leave extra space for the new variables introduced with the next element. Operations (2) and (3) can be applied to the M_{ij} and K_{ij} that are incompletely summed, as long as all of the M_{is} and K_{is} are completely summed, and as long as all contributions from later elements will eventually be added in. In practical terms, a node is "complete" (i.e., the coefficients of the variables at that node are fully summed) as soon as the last element tied to that node has been introduced. Consider Fig. 1: the elementary $[K]$ and $[M]$ for element A are added into the area reserved on core. Immediately, node 1 is complete and its variables can be eliminated, if they are slaves. Now the contributions from B are added in, completing nodes 2 and 3. At this stage, nodes 2–6, also any masters kept from node 1 are the variables represented on core. This pattern persists in a two-abreast problem: the variables on core are always five nodes and previous masters. The five nodes constitute a "front," which moves from one end of the structure to the other, embracing each node in turn. Irregular structures, including rings, etc., may have a front of variable width: the concept is quite general and can also be applied to equation-solving.

Assuming that the elements conform along their common boundaries, frequencies and buckling loads are overestimated, as in a pure stiffness method. The modal shape is expressed only in terms of the deflected shapes under unit master loads. Such constraints can be imposed, for sake of argument, by a large number of complex leverage systems that sense the nonconformity. If they are sprung to earth, the springs increasing by small steps towards infinite rate, a frequency increases by Rayleigh's principle. Conversely, if the nonconformity agitates a mass that increases to infinity, a frequency must come down. Therefore, one expects the estimated frequencies to interlace the true frequencies, apart from multiple roots or suppressed modes. Such bounds are of little practical interest, and it appears more important that a straightforward and efficient technique can be described for using meaningful generalized coordinates in a complicated structure, as suggested in Ref. 2, p. 56, for example.

References

[1] Irons, B., "Eigenvalue economisers in vibration problems," J. Roy. Aeronaut. Soc. 67, 526–528 (August 1963).

[2] Turner, M. J., "The direct stiffness method of structural analysis," AGARD Structures and Materials Panel Meeting, Aachen, Germany (September 17, 1959).